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Critical Point Theory and the Number of Solutions of a Nonlinear Dirichlet Problem

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Introduction and Summary.

The motivation for this paper stems from the following result:

THEOREM D. – *Let D be a bounded domain in R^n whose boundary ∂D is of class $C^{2+\alpha}$ for some $\alpha \in (0, 1)$. Let Δ denote the Laplacian and*

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$$

the sequence of eigenvalues of the boundary value problem

$$\begin{aligned} (\Delta u)(x) + \lambda u(x) &= 0 \qquad x \in D \\ u(x) &= 0 \qquad x \in \partial D\,, \end{aligned}$$

with each λ_n occurring in the sequence as often as its multiplicity. If $g \in C^1(R, R)$, and there exist constants γ and γ' and an integer N such that

$$\lambda_N < \gamma \leqslant g'(t) \leqslant \gamma' < \lambda_{N+1}$$

for all $t \in R$, then for any $p \in C^\alpha(\bar{D})$ there exists a unique solution of the boundary value problem

$$(P) \qquad \begin{aligned} \Delta u(x) + g\big(u(x)\big) &= p(x) \qquad x \in D \\ u(x) &= 0 \qquad\quad x \in \partial D\,. \end{aligned}$$

This result was essentially given by C. L. DOLPH in [11]. Although the result is not explicit in [11], it follows immediately from results concerning nonlinear integral equations of the Hammerstein type via use of the Green's function for the boundary value problem $(\Delta u)(x) = f(x)$, $x \in D$; $u(x) = 0$, $x \in \partial D$.

For different derivations which depend on implicit function theoretic arguments we refer the reader to the papers [9] and [16]. Generalizations, which give conditions for existence only, can be found in [13] and [14].

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AMBROSETTI and PRODI studied the boundary value problem (P) under the assumption that the range of g' contains an eigenvalue. Specifically, in [2], they showed that if $g(0) = 0$, $g''(t) > 0$ for all t, and $\lim_{t\to\pm\infty} g'(t) = g'(\pm\infty)$ with

$$0 < g'(-\infty) < \lambda_1 < g'(\infty) < \lambda_2\,,$$

then (P) has either zero, one, or two solutions. More precisely, they showed that in the Banach space $C^\alpha(\bar{D})$ there exists a C^2 closed manifold M whose complement consists of two components U_0 and U_2 such that (P) has no solution for $p \in U_0$, (P) has two solutions for $p \in U_2$, and (P) has one solution for $p \in M$.

In this paper we also consider the boundary value problem under the assumption that the range of g' contains an eigenvalue. We shall show that slight alterations of the conditions of Theorem D imply nonuniqueness of the solutions of (P) for suitably restricted $p(x)$. With stronger assumptions on g we can give the exact number of solutions. Specifically, we will prove the following results:

THEOREM A. – *Let D and p satisfy the same smoothness contitions as in Theorem D. Assume $g(0) = 0$, $g \in C^1(-\infty, \infty)$, and g' is bounded. If there exist an integer N and numbers γ and γ' such that $\lambda_N < \gamma < \gamma' < \lambda_{N+1}$ with $g'(t) \leq \gamma'$ for all $t \in (-\infty, \infty)$ and*

$$(*) \qquad -\infty < \inf_{t\in R}\left[\int_0^t g(s)\,ds - \frac{\gamma t^2}{2}\right],$$

and if

$$(**) \qquad g'(0) < \lambda_N\,,$$

then the homogeneous problem

$$(P_0) \qquad \begin{aligned} (\Delta u)(x) + g\big(u(x)\big) &= 0 \qquad x \in D \\ u(x) &= 0 \qquad x \in \partial D \end{aligned}$$

has at least two solutions; in particular, there exists a nontrivial solution of (P_0). If, in addition to $(**)$, we assume that

$$(***) \qquad g'(0) \neq \lambda_j \quad \textit{for all } j,$$

then, if the $L^2(D)$ norm of $p(x)$ is sufficiently small, the nonhomogeneous problem (P) has at least three solutions.

THEOREM B. – *Let p and D satisfy the same smoothness conditions as in Theorem D. Assume that $g(0) = 0$, $g \in C^2(R, R)$, and that $tg''(t) > 0$ almost everywhere. If $\lim_{t\to\infty} g'(t) = g'(\infty)$ and $\lim_{t\to\infty} g'(t) = g'(-\infty)$ are finite and there exists an integer N such that*

(*a*) $\lambda_{N-1} < g'(0) < \lambda_N$, (*b*) $\lambda_N < g'(\infty) < \lambda_{N+1}$,

and

(*c*) $\lambda_N < g'(-\infty) < \lambda_{N+1}$,

then there exists a number $r > 0$ such that problem (P) has exactly three solutions provided that the $L^2(D)$ norm of p is smaller than r. We emphasize that it is *not* necessary that $g'(\infty) = g'(-\infty)$.

In case the function g is odd we have the following substantial improvement of the first part of Theorem A:

THEOREM C. – *If the hypotheses of Theorem A are satisfied, if g is odd, and if $K \leqslant N$ is the integer such that*

$$\lambda_{K-1} \leqslant g'(0) < \lambda_K \leqslant \lambda_N$$

then there are at least $2(N-K+1)$ nontrivial solutions of (P_0).

In Theorem A, the condition $(*)$ will be satisfied if $\liminf_t g'(t) \geqslant \gamma$. However, $g'(t)$ can be less than γ for arbitrarily large values of t^2 and still satisfy condition $(*)$.

Our method of proving all three theorems consists of reducing a certain infinite dimensional problem to a finite dimensional problem and then applying finite dimensional critical point theory. Our main tool from critical point theory, which is Theorem 1 of the next section, appears to be new and we hope that it will have other applications. To prove Theorem C we make use of a result due to CLARK [8] concerning Lusternik-Schnirelman theory.

Although the idea of reducing problems such as P to finite dimensional problems has now become standard (see for example [3] or [6]), our method of reduction is novel in the sense that it involves a variational principle. The abstract methods developed in this paper can obviously be used to treat more general elliptic boundary value problems and nonlinear integral equations of the Hammerstein type—we have considered the simple problem (P) for clarity of exposition.

We mention one open problem related to Theorem C that we have not been able to resolve using our methods. In this case $n = 1$ we can show that, if the conditions of Theorem A hold, then the assertion of Theorem C is true *without the condition that g be odd.* We suspect that this also holds for $n > 1$.

1. – Finite dimensional critical point theory.

The main results of this section will be used only in the proof of the second assertion of Theorem A and in Theorem B.

Let $f \in C^1(R^n, R)$. If c is a real number we let

$$f^c = \{x \in R^n | f(x) \leqslant c\},$$

and

$$K_c = \{x \in R^n | f(x) = c,\ \nabla f(x) = 0\}\,.$$

We say that f satisfies the *Palais-Smale condtion*, or $(P - S)$ if, whenever $\{x_n\}_1^\infty$ is a sequence such that $\{f(x_n)\}_1^\infty$ is bounded and $\nabla f(x_n) \to 0$ as $n \to \infty$, then some subsequence of $\{x_n\}_1^\infty$ converges.

The following « deformation lemma » is a very special case of a known result due to CLARK [8].

LEMMA 1. – *If $f \in C^1(R^n, R)$ satisfies condition $(P - S)$, and if for some number c, $K_c = \Phi$, then there exists a number $\varepsilon > 0$ and a continuous function $F\colon R^n \times [0, 1] \to R^n$ such that:*

(a) $F(f^{c+\varepsilon}, 1) \subseteq f^{c-\varepsilon}$,

(b) $f(F(x, t)) \leqslant f(x)$ *for all* $x \in R^n$, $t \in [0, 1]$,

(c) $F(x, 0) = x$, $x \in R^n$.

In what follows we let $H_m(A)$ denote the m-th singular homology group over the integers of a topological space A; $H_m^{\#}(A)$ will denote the corresponding augmented homology group. We let $C_m(A)$ denote the group of singular chains on A; and if $z \in C_m(A)$ is m-cycle, with respect to the augmented boundary operator, then $[z] \in H_m^{\#}(A)$ will denote the augmented homology class of z. If $w \in C_m(A)$, then $|w| \subset A$ will denote the *support* of w. Finally, if $g\colon A \to B$ is continuous, then $C_m(g)\colon C_m(A) \to C_m(B)$ will denote the chain map induced by g, and $H_m^{\#}(g)\colon H_m^{\#}(A) \to H_m^{\#}(B)$ will denote the corresponding augmented homology functor. For further explanation of these terms see [12] or [22].

THEOREM 1. – *Let $f \in C^1(R^n, R)$ satisfy the $(P - S)$ condition. If for some number a and some integer $m \geqslant 0$, $H_m^{\#}(f^a) \neq \{0\}$, then there exists $\bar{x} \in R^n$ such that $f(\bar{x}) \geqslant a$ and $\nabla f(\bar{x}) = 0$*

PROOF. – We define a collection Σ of compact subsets of R^n as follows: Let z be any m-cycle on f^a, with respect to the augmented boundary operator, such that $[z] \neq 0$ in $H_m^{\#}(f^a)$. Clearly, z is also an (augmented) m-cycle on R^n. Therefore, since $H_m^{\#}(R^n) = \{0\}$, there exists $w \in C_{m+1}(R^n)$ such that $z = \partial_{m+1} w$, where $\partial_{m+1}\colon C_{m+1}(R^n) \to C_m(R^n)$ is the $m + 1$-dimensional boundary operator. We let Σ denote the collection of all supports $|w|$, where w is a singular $(m + 1)$-chain on R^n that arises in the foregoing manner. The hypothesis of the theorem implies that $\Sigma \neq \Phi$. We claim that

$$S \cap \{x | f(x) > a\} \neq \Phi \quad \text{if } S \in \Sigma. \tag{1}$$

Indeed, suppose $S = |w|$ where $\partial_{m+1} w = z \in C_m(f^a)$ and $[z] \neq 0$ in $H_m^{\#}(f^a)$. If (1) did not hold, then $w \in C_{m+1}(f^a)$ and hence $[z] = 0$ in $H_m^{\#}(f^a)$, which is a contradiction. Consequently,

$$c = \inf_{S \in \Sigma} \max_{x \in S} f(x) \geqslant a \,. \tag{2}$$

It follows that the theorem will be proved if we can show that $K_c \neq \Phi$. Let us assume that $K_c = \Phi$. By Clark's deformation lemma, there exists $F \colon R^n \times [0, 1] \to R^n$ such that assertions (a), (b) and (c) of Lemma 1 hold. If $f(x) \leqslant c$, then according to condition (b), $f\big(F(x, t)\big) \leqslant f(x) \leqslant c$ for $t \in [0, 1]$. Hence

$$F\big(f^c \times [0, 1]\big) \subset f^c \,. \tag{3}$$

Let $\overline{F}$ denote the restriction of F to $f^c \times [0, 1]$ and let $\overline{F}_k \colon f^c \to f^c$, $k = 0, 1$ be defined by $\overline{F}_0(x) = \overline{F}(x, 0)$ and $\overline{F}_1(x) = \overline{F}(x, 1)$. Considering the corresponding homology functors, $H_m^{\#}(\overline{F}_k) \colon H_m^{\#}(f_c) \to H_m^{\#}(f_c)$, we infer from (3) and the *homotopy invariance theorem* of homology theory (see [12, p. 45] or [22, p. 200]) that $H_m^{\#}(\overline{F}_0) = H_m^{\#}(\overline{F}_1)$. But, from condition (c) of Lemma 1, $\overline{F}_0(x) = x$ for $x \in f^c$, and hence

$$H_m^{\#}(\overline{F}_1) = Id \,. \tag{4}$$

If $\varepsilon > 0$ is as in condition (a) of Lemma 1, then by the definition of c in (2), there exists $S \in \Sigma$ such that $S \subseteq f^{c+\varepsilon}$. Consequently,

$$S_1 = F(S, 1) \subset f^{c-\varepsilon} \,. \tag{5}$$

Suppose $S = |w|$ with $\partial_{m+1} w = z \in C_m(f^a)$, and $[z] \neq 0$ in $H_m^{\#}(f^a)$. Let $F(\ , 1)(x) = F(x, 1)$ if $x \in R^n$. From (4), we have

$$\big[C_m\big(F(\ , 1)\big)\, z\big] = [C_m(\overline{F}_1)\, z] = H_m^{\#}(\overline{F}_1)\big([z]\big) = [z] \neq 0 \quad \text{ in } H_m^{\#}(f^c) \,.$$

Thus, since

$$\partial_{m+1} C_{m+1}\big(F(\ , 1)\big)\, w = C_m\big(F(\ , 1)\big)\, \partial_{m+1} w = C_m\big(F(\ , 1)\big)\, z \,,$$

it follows that $|C_{m+1}\big(F(\ , 1)\big)\, w| \in \Sigma$. Clearly,

$$|C_{m+1}\big(F(\ , 1)\big)\, w| = F\big(|w|, 1\big) = F(S, 1) = S_1 \,,$$

which proves that $S_1 \in \Sigma$. However, by (5) $\max_{x \in S_1} f(x) \leqslant c - \varepsilon$ wich contradicts the definitions of c given in (2). This contradiction proves that $K_c \neq \Phi$ and by earlier remarks it also proves the theorem.

In the following, if $v\colon R^n \to R^n$ is continuous and $\bar{x}$ is an isolated zero of v, then $i(v, \bar{x})$ will denote the index of v at $\bar{x}$ (see [17, p. 32]).

THEOREM 2. – *Let $f \in C^2(R^n, R)$. If $f(x) \to \infty$ as $\|x\| \to \infty$ and the set of solutions of $\nabla f(x) = 0$ is a finite set $\{x_0, x_1, x_2, \ldots, x_k\}$, then*

$$\sum_{j=0}^{k} i(\nabla f, x_j) = 1 .$$

PROOF. – Clearly, the growth condition on f implies that f satisfies $(P - S)$. If the number a is chosen so large $f(x_j) < a$ for $j = 0, 1, \ldots, k$ then, by Theorem 1, we infer that $H^{\#}_m(f^a) = \{0\}$ for all $m \geqslant 0$. Hence, $H_m(f^a) = \{0\}$ for $m \geqslant 1$, and $H_0(f^a)$ is infinite cyclic. Since a is a regular value for f, f^a is a C^1 n-manifold with boundary and the C^1 vector field ∇f points outward at each boundary point. Since $x_j \in f^a$, $j = 0, \ldots, k$, the Poincaré-Hopf theorem ([17, p. 35]) implies that

$$\sum_{j=0}^{k} i(\nabla f, x_j) = \sum_{j=0}^{n} (-1)^j \operatorname{rank}\big(H_j(f^a)\big) = 1$$

and the theorem is proved.

If $\nabla f(\bar{x}) = 0$, we say that $\bar{x}$ is a nondegenerate critical point of f if the Hessian matrix of f at $\bar{x}$ is nonsingular.

THEOREM 3. – *Let $f \in C^2(R^n, R)$. If $f(x) \to \infty$ as $\|x\| \to \infty$; if $\min_{x \in R^n} f(x) = f(x_0)$ and if there exists a nondegenerate critical point x_1 of f such that $x_1 \neq x_0$, then f has at least three distinct critical points.*

PROOF. – If x_0 is a point such that $f(x_0) = \min_{x \in R^n} f(x)$, then $\nabla f(x_0) = 0$, and if f has an isolated minimum at x_0, then $i(\nabla f, x_0) = 1$. A proof of this geometrically evident fact can be found in [20] or [21]. Since x_1 is a nondegenerate critical point of f, it follows that $i(\nabla f, x_1) = \pm 1$ (see [17, p. 37]). If the theorem were false the only critical points of f would be x_1 and an isolated minimum point x_0. Consequently, the sum of the indices of ∇f at its zeros would be 0 or 2, contradicting Theorem 2. This proves the result.

REMARK. – Using the reasoning of Theorem 3, RABINOWITZ [21] proved that if B is a bounded domain, with $\nabla f \neq 0$ on the boundary; if the Brouwer degree $d(\nabla f, B, 0) = 1$; if f has a local minimum at a point $x_0 \in B$; and if f has a nondegenerate critical point $x_1 \in B$ with $x_1 \neq x_0$; then f has at least three critical points in B. Assuming that f satisfies the hypotheses of Theorem 2, the proof of Theorem 2 shows that if $B = \{x | f(x) < a\}$, then $d(\nabla f, B, 0) = 1$ provided a is large.

2. – An abstract theorem on Hilbert space functionals.

Our next result is a strengthening of a theorem proved in [5] and [16]. A few details will be omitted in the proof.

Let H be a real Hilbert space and F a real valued function defined on H with a second continuous Fréchet derivative. As is customary, we define a C^1 map ∇F: $H \to H$ such that $F'(u)(w) = \langle \nabla F(u), w \rangle$ by means of the Riesz-Frechet theorem. The derivative of ∇F at $u \in H$, which is a self-adjoint operator on H, will be denoted by $D^2F(u)$.

THEOREM 4. – *Let $F \in C^2(H, R)$, and suppose the following conditions are satisfied:*

(*a*) *$\nabla F(0) = 0$ and there exist closed subspaces X_1 and Y_1 of H and a constant $m_1 > 0$ such that*

(i) $H = X_1 \oplus Y_1$,

(ii) $\dim X_1 < \infty$,

(iii) $\langle D^2F(0)x, x\rangle \leqslant 0$, *for all* $x \in X_1$,

(iv) $\langle D^2F(0)y, y\rangle \geqslant m_1\|y\|^2$ *for all* $y \in Y_1$.

(*b*) *There exist closed subspaces X and Y of H and a constant $m > 0$ such that*

(v) $H = X \oplus Y$,

(vi) $\dim X_1 < \dim X < \infty$,

(vii) $(F|X)(x) \to -\infty$ *as* $\|x\| \to \infty$ (*where $F|X$ is the restriction of F to X*),

(viii) $\langle D^2F(u)y, y\rangle \geqslant m\|y\|^2$ *for all $y \in Y$ and all $u \in H$.*

ASSERTION. – *There exists $u_0 \in H$ with $u_0 \neq 0$ such that $\nabla F(u_0) = 0$. Moreover,*

$$F(u_0) = \max_{x \in X} \min_{y \in Y} F(x + y)\,.$$

If condition (iii) *is replaced by*

(iii*) $\langle D^2F(0)x, x\rangle < 0$ *if $x \in X_1$ and $x \neq 0$,*
there exists $u_2 \in H$ with $u_2 \neq 0$ and $u_2 \neq u_0$ such that $\nabla F(u_2) = 0$.

PROOF. – For fixed $\hat{x} \in X$ define g: $Y \to R$ by $g(y) = F(\hat{x} + y)$. If $k \in Y$ then

$$\langle \nabla g(y), k\rangle = \frac{d}{dt} g(y + tk)|_{t=0} = \langle \nabla F(\hat{x} + y), k\rangle$$

and

$$\langle D^2 g(y)k, k\rangle = \frac{d^2}{dt^2} g(y + tk)|_{t=0} = \langle D^2F(\hat{x} + y)k, k\rangle\,.$$

Hence, by (viii), for $y \in Y$ and $k \in Y$,

$$\langle D^2 g(y)\, k, k\rangle \geqslant m\|k\|^2 . \tag{6}$$

As is well-known (see, for example [23, p. 79-80]), (6) implies the existence of $\hat{y} \in K$ such that

$$g(\hat{y}) = \min_{y\in Y} g(y) \qquad \text{and} \qquad \nabla g(\hat{y}) = 0 .$$

Since, for some $s \in (0, 1)$,

$$\langle \nabla g(y_1) - \nabla g(y_2), y_1 - y_2\rangle = \langle D^2 g(y_2 + s(y_1 - y_2))(y_1 - y_2), y_1 - y_2\rangle \geqslant m\|y_1 - y_2\|^2$$

(see [23, p. 37]), ∇g can have only one zero on Y so $\hat{y}$ is unique. Setting $\hat{y} = \varphi(\hat{x})$ this defines a map $\varphi\colon X \to Y$. Thus, given $x \in X$, $\varphi(x)$ is the unique member of Y such that

$$\langle \nabla F(x + \varphi(x)), k\rangle = 0 \qquad \text{for all } k \in Y, \tag{7}$$

and such that

$$F(x + \varphi(x)) = \min_{y\in Y} F(x + y) . \tag{8}$$

A simple argument based on the implicit function theorem shows that the condition (viii) implies that φ is of class C^1. See [16, p. 597-598] for the details.

We claim that the function $G\colon X \to R$, which is defined by $G(x) = F(x + \varphi(x))$, is of class C^2. This is not immediately apparent since we only know that $\varphi \in C^1(Y, X)$. If $h \in X$, we have

$$\langle \nabla G(x), h\rangle = \frac{d}{dt} G(x + th)|_{t=0} = \frac{d}{dt} F(x + th + \varphi(x + th))|_{t=0} = \\ = \langle \nabla F(x + \varphi(x)), h + \varphi'(x)(h)\rangle .$$

Since $\varphi'(x)$ is a linear map from X to Y we see that $k = \varphi'(x)(h) \in Y$, so by (7) and the above

$$\langle \nabla G(x), h\rangle = \langle \nabla F(x + \varphi(x)), h\rangle . \tag{9}$$

Since F is of class C^2, ∇F is of class C^1; hence, since $\varphi \in C^1$, it follows that $\langle \nabla G(x), h\rangle$ is of class C^1 for all $h \in K$. This implies that G is of class C^2, and if $h \in X$,

$$\langle D^2 G(x) h, h\rangle = \frac{d}{dt}\langle \nabla G(x + th), h\rangle|_{t=0} = \frac{d}{dt}\langle \nabla F(x + th + \varphi(x + th)), h\rangle|_{t=0} \\ = \langle D^2 F(x + \varphi(x))(h + \varphi'(x)(h)), h\rangle . \tag{10}$$

In order to obtain another expression for $\langle D^2G(x)h, h\rangle$ we observe from (7) that if $h\in X$ and $k\in Y$ then $\langle \nabla F(x+th+\varphi(x+th)), k\rangle = 0$ for all t.

Hence,

$$\frac{d}{dt}\langle \nabla F(x+th+\varphi(x+th)), k\rangle|_{t=0} = \\ = \langle D^2F(x+\varphi(x))(h+\varphi'(x)(h)), k\rangle = 0 \quad \text{if } k\in Y\,. \tag{11}$$

Therefore, by setting $k=\varphi'(x)(h)\in Y$, we see from (10) that, if $h\in X$, then

$$\langle D^2G(x)h, h\rangle = \langle D^2F(x+\varphi(x))(h+\varphi'(x)(h)), h+\varphi'(x)(h)\rangle\,. \tag{12}$$

To prove the first assertion of Theorem 4, we observe, by setting $y=0$ in (8), that

$$F(x+\varphi(x))\leqslant F(x) \quad \text{if } x\in X.$$

Hence, according to assumption (viii),

$$G(x) = F(x+\varphi(x)) \to -\infty \quad \text{as } \|x\|\to\infty. \tag{13}$$

Since $\dim X<\infty$, this implies the existence of $x_0\in X$ with

$$G(x_0) = \max_{x\in X} G(x)\,. \tag{14}$$

Therefore, if $h\in X$ is arbitrary, we infer from (9) that $\langle\nabla G(x_0), h\rangle = \langle \nabla F(x_0 + \varphi(x_0)), h\rangle = 0$. Since $H = X\oplus Y$, if $w\in H$, $w=h+k$ with $h\in X$ and $k\in Y$, so by (7) and the above $\langle\nabla F(x_0+\varphi(x_0)), w\rangle = 0$. Consequently, if $u_0 = x_0+\varphi(x_0)$ then $\nabla F(u_0)=0$, and from (8) and (14) it follows that

$$F(u_0) = \max_{x\in X} F(x+\varphi(x)) = \max_{x\in X}\min_{y\in Y} F(x+y)\,.$$

To complete the proof of the first assertion we must show that $u_0\neq 0$. To this end we consider the subspace W of H defined by

$$W = \{w\in H | w = h+\varphi'(x_0)(h),\ h\in X\}\,.$$

Since $\varphi'(x_0)(h)\in Y$ if $h\in X$ and $X\cap Y = \{0\}$, it follows that $\dim W = \dim X$. If $h\in X$, then by (14),

$$\langle D^2G(x_0)h, h\rangle = \frac{d^2}{dt^2} G(x_0+th)|_{t=0}\leqslant 0\,.$$

Hence, using (12), we see that

$$\langle D^2F(x_0+\varphi(x_0))(h+\varphi'(x_0)(h)),\ h+\varphi'(x_0)(h)\rangle \leq 0$$

for all $h \in X$, and so

$$\langle D^2F(u_0)w, w\rangle \leq 0 \quad \text{for } w \in W. \tag{15}$$

Let $w_1, \ldots, w_m$ be a basis for W. According to (i) $H = X_1 \oplus Y_1$ so there exist $r_k \in X_1$ and $s_k \in Y_1$ for $k = 1, \ldots, m$ such that

$$w_k = r_k + s_k\,, \qquad k = 1, \ldots, m.$$

By (vi), $\dim X_1 < \dim X = \dim W = m$, so there exist numbers $c_1, \ldots, c_m$, not all zero, such that $c_1 r_1 + \ldots + c_m r_m = 0$; therefore

$$\hat{w} = \sum_{k=1}^{m} c_k w_k = \sum_{k=1}^{m} c_k s_k \in Y_1\,,$$

and $\hat{w} \neq 0$. Thus by condition (iv), $\langle D^2F(0)\hat{w}, \hat{w}\rangle > 0$. Since $\hat{w} \in W$, it follows from (15) that $u_0 \neq 0$, and the proof of the first part of Theorem 4 is complete.

To prove the second assertion of Theorem 4, we first observe that 0 is a critical point of G distinct from x_0. Indeed, since $\nabla F(0) = 0$, and since according to (6), given $x \in X$, $\varphi(x)$ is the unique member of Y such that $\langle \nabla F(x+\varphi(x)), k\rangle = 0$ for all $k \in Y$, it follows that $\varphi(0) = 0$. Therefore, by (9), if $h \in X$, $\langle \nabla G(0), h\rangle = \langle \nabla F(0 + \varphi(0)), h\rangle = \langle \nabla F(0), h\rangle = 0$, so $\nabla G(0) = 0$. Since, as shown above $u_0 = x_0 + \varphi(x_0) \neq 0$, it follows that $x_0 \neq 0$.

We claim that the condition (iii*) implies that 0 is a nondegenerate critical point. To see this, we first show that the kernel of $D^2F(0)$ is trivial. If $D^2F(0)u = 0$ and $u = r + s$ with $r \in X_1$ and $s \in Y_1$ then the self-adjointness of $D^2F(0)$ implies that

$$0 = \langle r - s, D^2F(0)(r+s)\rangle = \langle r, D^2F(0)r\rangle - \langle s, D^2F(0)s\rangle\,.$$

Since conditions (iii*) and (iv) imply that $\langle s, D^2F(0)s\rangle < 0$ unless $s = 0$, and $\langle r, D^2F(0)r\rangle > 0$ unless $r = 0$, it follows that $u = 0$. Suppose now that for some $h_1 \in X$ $D^2G(0)h_1 = 0$. It follows from (9) that if $h_2 \in X$ then

$$\begin{aligned} 0 = \langle D^2G(0)h_1, h_2\rangle &= \frac{d}{dt}\langle \nabla G(th_1), h_2\rangle\big|_{t=0} \\ &= \frac{d}{dt}\langle \nabla F(th_1 + \varphi(th_1)), h_2\rangle\big|_{t=0} = \langle D^2F(0)(h_1 + \varphi'(0)h_1), h_2\rangle = 0\,. \end{aligned} \tag{16}$$

From (11) we see that if $k \in Y$ then

$$\langle D^2F(0)(h_1 + \varphi'(0)h_1), k\rangle = \langle D^2F(0 + \varphi(0))(h_1 + \varphi'(0)h_1), k\rangle = 0 .$$

Since $H = X \oplus Y$, this and (16) show that $\langle D^2F(0)(h_1 + \varphi'(0)h_1), u\rangle = 0$ for all $u \in H$. Hence $h_1 + \varphi'(0)h_1 = 0$ so $h_1 = 0$. This proves the claim that the linear map $D^2G(0): X \to X$ is nonsingular.

Consider the function $f: X \to R$ defined by $f(x) = -G(x)$. By (13) and (14), $f(x) \to \infty$ as $\|x\| \to \infty$ and $f(x_0) = \min_{x \in X} f(x)$. As shown above 0 is a nondegenerate critical point of f distinct from x_0. Therefore, by Theorem 3 of the previous section, there exists a point $x_2 \in X$, with $x_2 \neq x_0$ and $x_2 \neq 0$ such that $\nabla f(x_2) = 0$. Therefore if $h \in X$, it follows from (9) that $\langle \nabla F(x_2 + \varphi(x_2)), h\rangle = \langle \nabla G(x_2), h\rangle = 0$. Since according to (7) $\langle \nabla F(x_2 + \varphi(x_2)), k\rangle = 0$ for all $k \in Y$, we infer that $\nabla F(x_2 + \varphi(x_2)) = 0$. Finally, since $X \cap Y = \{0\}$, we see that $u_2 = x_2 + \varphi(x_2) \neq x_0 + \varphi(x_0) = u_0$, and $u_2 \neq 0$. This completes the proof of Theorem 4.

3. – Proof of Theorem A.

We shall first prove Theorems A and B for weak solutions and defer discussion of regularity until the end of the paper.

To derive Theorem A from Theorem 4 we take as our real Hilbert space H the Sobolev space $\mathring{H}_1(D)$ which is the completion of the inner product space consisting of real C^1 functions having support contained in D with inner product

$$\langle u, v\rangle_1 = \int_D \langle \nabla u(x), \nabla v(x)\rangle\, dx .$$

We let $\langle\,,\,\rangle_0$ denote the usual $L^2(D)$ inner product. If $\{\lambda_m\}_1^\infty$ is the sequence defined in the introductory section, if for each m, φ_m satisfies $(\Delta\varphi_m)(x) + \lambda_m\varphi_m(x) = 0$ if $x \in D$ and $\varphi_m(x) = 0$ for $x \in \partial D$, and if $\langle\varphi_m, \varphi_l\rangle_0 = \delta_{ml}$, then for all $u \in H$

$$\langle u, u\rangle_1 = \sum_{m=1}^{\infty} \lambda_m \langle\varphi_m, u\rangle_0^2 \tag{17}$$

and

$$\langle u, u\rangle_0 = \sum_{m=1}^{\infty} \langle\varphi_m, u\rangle_0^2 \tag{18}$$

(see, for example, [9] or [4]). If N is the integer that appers in the statement of Theorem A, we let X denote the finite dimensional subspace of H spanned by φ_1, $\varphi_2, \dots, \varphi_N$ and $Y = X^\perp$. From (17) and (18) it follows that

$$\langle y, y\rangle_1 \geq \lambda_{N+1}\langle y, y\rangle_0 \quad \text{if } y \in Y, \tag{19}$$

and

$$\langle x, x\rangle_1 \leqslant \lambda_N \langle x, x\rangle_0 \quad \text{if } x \in X. \tag{20}$$

Let $K \leqslant N$ be the integer such that

$$\lambda_{K-1} \leqslant g'(0) < \lambda_K \leqslant \lambda_N , \tag{21}$$

where $\lambda_0 = -\infty$, and let X_1 denote the span of $\{\varphi_1, \dots, \varphi_{K-1}\}$ and $Y_1 = X_1^\perp$. We define $F\colon H \to R$ by

$$F(u) = \frac{\langle u, u\rangle_1}{2} - \int_D V(u(x))\, dx \tag{22}$$

where $V(t) = \int_0^t g(s)\, ds$. As is shown in [16], the boundedness and continuity of g' implies that $F \in C^2(H, R)$. Moreover, if $w \in H$

$$\langle \nabla F(u), w\rangle_1 = \frac{d}{dt} F(u + tw)|_{t=0} = \langle u, w\rangle_1 - \int_D g(u(x))\, w(x)\, dx .$$

Therefore, weak solutions of the boundary value problem (P_0) coincide with critical points of F.

If u, v, and w are in H, then

$$\langle D^2F(u)v, w\rangle_1 = \frac{d}{dt}\langle \nabla F(u + tv), w\rangle_1|_{t=0} = \langle v, w\rangle_1 - \int_D g'(u(x))\, v(x) w(x)\, dx . \tag{23}$$

Thus from (19) and the hypothesis of Theorem A, we see that if $y \in Y$,

$$\langle D^2F(u)y, y\rangle_1 \geqslant \langle y, y\rangle_1 - \gamma' \langle y, y\rangle_0 \geqslant m\langle y, y\rangle_1 , \qquad m = 1 - \frac{\gamma'}{\lambda_{N+1}} > 0 . \tag{24}$$

Hence condition (viii) of Theorem 4 is satisfied. If $x \in X$, it follows from condition (*) of Theorem A, (20), and (22) that for some constant c

$$F(x) \leqslant \frac{\langle x, x\rangle_1}{2} - \frac{\gamma\langle x, x\rangle_0}{2} + c \leqslant \frac{1}{2}\left(1 - \frac{\gamma}{\lambda_N}\right)\langle x, x\rangle_1 + c .$$

Since $\lambda_N < \gamma$ we see that condition (vii) of Theorem 4 si satisfied.

From the definition of X_1 and Y_1, and (17)-(18), we have $\langle r, r\rangle_1 \leqslant \lambda_{K-1}\langle r, r\rangle_0$ if $r \in X_1$ and $\langle s, s\rangle_1 \geqslant \lambda_K \langle s, s\rangle_0$ if $s \in Y_1$. Consequently, from (21) and (23) we see that, if $r \in X_1$, then

$$\langle D^2F(0)r, r\rangle_1 = \langle r, r\rangle_1 - \int_D g'(0)\, r(x)^2\, dx \leqslant \langle r, r\rangle_1 - \lambda_{K-1}\langle r, r\rangle_0 \leqslant 0 ;$$

while, if $s \in Y_1$, then

$$\langle D^2F(0)s, s\rangle_1 = \langle s, s\rangle_1 - g'(0)\langle s, s\rangle_0 \geq \left(1 - \frac{g'(0)}{\lambda_K}\right)\langle s, s\rangle_1 = m_1\langle s, s\rangle_1 .$$

Since $m_1 > 0$, conditions (iii) and (iv) of Theorem 4 are verified. Since the remainder of the conditions (i)-(viii) of Theorem 4 are obviously satisfied, it follows that conditions (*) and (**) of Theorem A imply the existence of at least two solutions of problem (P_0).

Suppose now that condition (***) of Theorem A is satisfied. In this case we see from (23) that

$$\lambda_{K-1} < g'(0) < \lambda_K . \tag{25}$$

If X_1 and Y_1 are defined as before, then the inequality $\langle D^2F(0)s, s\rangle_1 \geq m_1\langle s, s_1\rangle_1$ for $s \in Y_1$ is still valid and our previous reasoning shows that if $r \in X_1$, then

$$\langle D^2F(0)r, r\rangle_1 = \langle r, r\rangle_1 - g'(0)\langle r, r\rangle_0 \leq \left[1 - \frac{g'(0)}{\lambda_{K-1}}\right]\langle r, r\rangle_1 = -m_2\langle r, r\rangle_1$$

where $m_2 > 0$. Since D^2F is continuous there exists $\delta_1 > 0$ such that $|D^2F(u) - D^2F(0)| < \min(m_1/2, m_2/2)$ if $|u| < \delta_1$. Hence, by the above

$$\langle D^2F(u)r, r\rangle_1 \leq -\frac{m_2}{2}\langle r, r\rangle_1 , \qquad r \in X_1 , \tag{26}$$

$$\langle D^2F(u)s, s\rangle_1 \geq \frac{m_1}{2}\langle s, s\rangle_1 , \qquad s \in Y_1 , \tag{27}$$

if $|u| < \delta_1$.

From (23), if $v, w \in H$, $\langle D^2F(0)v, w\rangle_1 = \langle v, w\rangle_1 - g'(0)\langle v, w\rangle_0$; hence, $D^2F(0)v = v - g'(0)Tv$, where T is the linear operator on H defined by $\langle Tv, w\rangle_1 = \langle v, w\rangle_0$. Since the injection from $\mathring{H}_1(D)$ into $L^2(D)$ is compact, T is a compact operator on H. If $u \in$ kernel $D^2F(0)$, then u is a weak solution of $\Delta u + g'(0)u = 0$ in D, $u = 0$ on ∂D. Since, by standard regularity theory, u is a classical solution, we see from (25) that $u = 0$. Hence, by the Fredholm alternative, the continuous linear map $D^2F(0)$ $D^2F(0)\colon H \to H$ is one-to-one and onto. Therefore, since $\nabla F(0) = 0$, it follows from the inverse function theorem (see, for example [15]) that there exists an open set U in H containing 0 such that the restriction of ∇F to U is one-to-one, $\nabla F(U)$ is an open set containing 0, and ∇F restricted to U has a C^1 inverse. Without loss of generality we may assume that $|u|_1 < \delta_1$ for all $u \in U$.

Suppose that $r > 0$ is such that $|v|_1 < r$ implies that $v \in \nabla F(U)$. We claim that if $p \in L^2(D)$ and $|p|_0 < \sqrt{\lambda_1}\, r$, then there exists a unique weak solution φ of the problem $\Delta u + g(u) = p(x)$, $u = 0$ on ∂D such that $|\varphi|_1 < \delta_1$. To see this we note that $-\langle p, w\rangle_0$, $w \in \mathring{H}_1(D)$, represents a continuous linear functional on H so according to the Riesz representation theorem there exists $v \in \mathring{H}_1(D)$ such that $-\langle p, w\rangle_0 = \langle v, w\rangle_1$. By the Schwarz inequality, (17), and (18)

$$|v|_1^2 \leqslant |p|_0 |v|_0 \leqslant |p|_0 \frac{1}{\sqrt{\lambda_1}} |v|_1 ;$$

hence $|v|_1 < r$. Therefore there exists φ with $|\varphi|_1 < \delta_1$ such that $\nabla F(\varphi) = v$. Consequently, for $w \in \mathring{H}_1(D)$, $\langle \nabla F(\varphi), w\rangle_1 = \langle v, w\rangle_1 = -\langle p, w\rangle_0$ or, by (22)

$$\int_D (\langle \nabla\varphi, \nabla w\rangle - g(\varphi)w + pw)\,dx = 0 . \tag{28}$$

This shows that φ is a weak solution of $\Delta u + g(u) = p$.

We now fix p and φ and define $F_1\colon H \to R$ by

$$F_1(u) = \int_D \left(\frac{\langle \nabla(u+\varphi), \nabla(u+\varphi)\rangle}{2} - V(u+\varphi) + p[u+\varphi]\right) dx .$$

We will show that F_1 satisfies the conditions of Theorem 4, with (iii) replaced by (iii*). Since for $w \in \mathring{H}_1(D)$

$$\langle \nabla F_1(u), w\rangle_1 = \frac{d}{dt} F_1(u+tw)|_{t=0} = \int_D (\langle \nabla(u+\varphi), \nabla w\rangle - g(u+\varphi)w + pw)\,dx$$

we see that $\nabla F_1(0) = 0$ and that $\nabla F_1(u) = 0$ if and only if $u + \varphi$ is a weak solution of (P). Moreover, since

$$\langle D^2F_1(u)v, w\rangle_1 = \frac{d}{dt}\langle \nabla F_1(u+tv), w\rangle_1|_{t=0} = \int_D (\langle \nabla v, \nabla w\rangle - g'(u+\varphi)vw)\,dx ,$$

it follows from (23) that $D^2F_1(u) = D^2F(u+\varphi)$. Consequently, if $y \in Y$, we see from (24) that

$$\langle D^2F(u)y, y\rangle_1 \geqslant m\langle y, y\rangle_1$$

for all $u \in H$. This shows that condition (viii) of Theorem 4 holds for F_1. Since $|\varphi|_1 < \delta_1$, we see from (26) and (27) that, if $r \in X_1$, then

$$\langle D^2F_1(0)r, r\rangle_1 = \langle D^2F(\varphi)r, r\rangle_1 \leqslant -\frac{m_2}{2}\langle r, r\rangle_1 ,$$

while if $s \in Y_1$,

$$\langle D^2F_1(0)s, s\rangle_1 = \langle D^2F(\varphi)s, s\rangle_1 \geqslant \frac{m_1}{2}\langle s, s_1\rangle_1 .$$

Thus F_1 satisfies conditions (iii*) and (iv) of Theorem 4.

To see that F_1 satisfies condition (vii) of Theorem 4, let $x \in X$. From (20) and (*) of Theorem *A*, we see that for some constants c and c'

$$\begin{aligned} F_1(x) &= \frac{\langle x+\varphi, x+\varphi\rangle_1}{2} - \int_D V(x(\xi)+\varphi(\xi))\,d\xi + \langle p, x\rangle_0 \\ &\leqslant \frac{1}{2}|x+\varphi|_1^2 - \frac{\gamma}{2}|x+\varphi|_0^2 + c + |p|_0|x|_0 \\ &\leqslant \frac{1}{2}\left[|x|_0^2 - \gamma|x|_0^2\right] + |\varphi|_1|x|_1 + \left(|p|_0 + \gamma|\varphi|_0\right)|x|_0 + c' \\ &\leqslant \frac{1}{2}\left(1-\frac{\gamma}{\lambda_N}\right)|x|_1^2 + |\varphi|_1|x|_1 + \frac{1}{\sqrt{\lambda_1}}\left(|p|_0 + \gamma|\varphi|_0\right)|x|_1 + c'\,. \end{aligned}$$

Hence $(F_1|X)(x) \to -\infty$ as $|x|_1 \to \infty$ and F_1 satisfies condition (vii). Since the remaining conditions of Theorem 4 obviously hold, it follows from the second assertion of Theorem 4 that there exist u_0 and u_2 with $u_0 \neq u_2$, $u_0 \neq 0$, $u_2 \neq 0$ such that $\nabla F_1(\varphi + u_k) = 0$, $k = 0, 2$. By the above remarks, φ, $\varphi + u_0$, and $\varphi + u_2$ are distinct weak solutions of problem (P).

4. – Proof of Theorem *B*.

We shall prove Theorem *B* via several lemmas. The first, which is stated for future reference, is essentially implicit in the proof of Theorem 4.

LEMMA 1. – *Let H be a real Hilbert space and let $F \in C^2(H, R)$ satisfy conditions* (v), (vii), *and* (viii) *of Theorem* 4 *with* $\dim X < \infty$. *In order that $\nabla F(u) = 0$ it is necessary and sufficient that for some $x \in X$, $u = x + \varphi(x)$ and $\nabla G(x) = 0$ where $\varphi\colon X \to Y$ and $G\colon X \to R$ are defined by* (8) *and* (13) *respectively.*

The sufficiency of these conditions follows immediately from (7), (9), and condition (v). As shown in the proof of Theorem 4, if for a given $\hat{x} \in X$ there exists a $y \in Y$ such that $\langle \nabla F(\hat{x} + y), k\rangle = 0$ for all $k \in Y$, then $y = \varphi(\hat{x})$. Therefore, the necessity is clear from (v) and (9).

Assume that the function g satisfies the conditions of Theorem *B*. From the condition $tg''(t) > 0$, a.e. and conditions (a), (b) and (c) it is clear that g satisfies the hypotheses of the first part of Theorem *A*, where we may choose γ and γ' to be any numbers satisfying

$$\lambda_N < \gamma < \min\{g'(-\infty), g'(\infty)\} \leqslant \max\{g'(-\infty), g'(\infty)\} \leqslant \gamma' < \lambda_{N+1}\,.$$

Moreover, from (a), g satisfies condition (***) of Theorem *A*. Therefore, referring to the proof of Theorem A and letting the functions $\{\varphi_m\}_1^\infty$ have the same meaning as before, we have

LEMMA 2. – *Let g satisfy the conditions of Theorem B. If $H = \mathring{H}_1(D)$ and F is defined as in* (22) *then F satisfies conditions* (i), (ii), (iii*), (iv)-(viii) *of Theorem* 4 *where*

$$X = \operatorname{span}\{\varphi_1, \ldots, \varphi_N\}, \qquad Y = X^\perp$$

and

$$X_1 = \operatorname{span}\{\varphi_1, \ldots, \varphi_{N-1}\}, \qquad Y_1 = X_1^\perp .$$

LEMMA 3. – *Assume that the hypotheses of Theorem B hold. Let F be defined as in Lemma 2 and suppose that $\nabla F(u_0) = 0$ so, by Lemma 1, $u_0 = x_0 + \varphi(x_0)$ with $x_0 \in X$ and $\nabla G(x_0) = 0$. If $u_0 \neq 0$ then* $\operatorname{sgn}\det D^2G(x_0) = (-1)^N$ *and* $\operatorname{sgn}\det D^2G(0) = (-1)^{N-1}$.

PROOF. – Suppose $\nabla F(u_0) = 0$ and $u_0 \neq 0$. As shown in the proof of Theorem A,

$$\Delta u_0 + g(u_0) = 0 \quad \text{in } D, \qquad u_0|\partial D = 0 \tag{29}$$

in the weak sense. Since $g(0) = 0$ we may rewrite (29) in the form

$$\Delta u_0 + q(x)\, u_0 = 0 \quad \text{in } D, \qquad u_0|\partial D = 0 \tag{30}$$

where

$$q(x) = \int_0^1 g'\big(su_0(x)\big)\, ds . \tag{31}$$

In the last section it will be shown that, as a result of standard regularity theory, u_0 is actually a classical solution of (30). Assuming that this is true for the time being, the simple form of (30) and known results on unique continuation ([4, p. 160-163], [18, p. 59-61]) imply that u_0 *cannot vanish identically on an open subset of D.*

We consider the two eigenvalue problems

$$\Delta w + \alpha q(x)\, w = 0 \quad \text{in } D, \qquad w|\partial D = 0 \tag{32}$$

and

$$\Delta w + \beta g'\big(u_0(x)\big)\, w = 0 \quad \text{in } D, \qquad w|\partial D = 0 . \tag{33}$$

Let

$$\alpha_1 < \alpha_2 \leqslant \ldots \leqslant \alpha_k \leqslant \alpha_{k+1} \leqslant \ldots$$

and

$$\beta_1 < \beta_2 \leqslant \ldots \leqslant \beta_k \leqslant \beta_{k+1} \leqslant \ldots$$

denote the eigenvalues of the problems (32) and (33) respectively, with each eigenvalue occuring as often as the number of independent solutions associated with it. We claim that

$$\beta_N < 1 < \beta_{N+1}\,, \tag{34}$$

where N is the integer which occurs in the statement of Theorem B. To see this, let $\{\psi_k\}_{k=1}^{\infty}$ and $\{\theta_k\}_{k=1}^{\infty}$ be sequences of functions in $\mathring{H}_1(D)$ such that

$$\Delta\psi_k + \alpha_k q(x)\,\psi_k = 0\,, \qquad \Delta\theta_k + \beta_k g'(u_0(x))\,\theta_k = 0$$

for all k, and such that

$$\int_D q(x)\,\psi_k(x)\,\psi_j(x)\,dx = \delta_{kj}\,, \qquad \int_D g'(u_0(x))\theta_k(x)\,\theta_j(x)\,dx = \delta_{kj}\,.$$

For fixed k, choose numbers $c_1, \ldots, c_k$ not all zero such that if $v = \sum_{j=1}^{k} c_j\psi_j$ then

$$\int_D g'(u_0(x))\,v(x)\,\theta_j(x)\,dx = 0\,, \qquad j = 1, \ldots, k-1.$$

According to the Rayleigh quotient characterization of the eigenvalues of (33) ([9]) we have

$$\beta_k \leqslant \int_D \langle\nabla v, \nabla v\rangle\,dx \Big/ \int_D g'(u_0)\,v^2\,dx\,. \tag{35}$$

Since $tg''(t) > 0$ for almost all $t \in (-\infty, \infty)$, we see that if $x \in D$ and $u_0(x) \neq 0$, then

$$q(x) = \int_0^1 g'(su_0(x))\,ds < g'(u_0(x))\,.$$

Therefore, since $u_0(x)$ cannot vanish identically on any open subset of D, we see from (35) that

$$\beta_k < \int_D \langle\nabla v, \nabla v\rangle\,dx \Bigg/ \int_D q(x)\,v^2\,dx = \sum_{j=1}^{k}\alpha_j c_j^2 \Big/ \sum_{j=1}^{k} c_j^2\,,$$

and hence

$$\beta_k < \alpha_k\,, \qquad k = 1, 2, \ldots\,. \tag{36}$$

We now consider the two eigenvalue problems

$$\Delta w + \gamma\lambda_{N+1} w = 0 \quad \text{in } D, \qquad w|\partial D = 0 \tag{37}$$

and

$$\Delta w + \delta\lambda_{N+1} w = 0 \quad \text{in } D\,, \qquad w|\partial D = 0\,. \tag{38}$$

If $\{y_k\}_1^\infty$ and $\{\delta_k\}_1^\infty$ denote the eigenvalues of (37) and (38) respectively, indexed in order of increasing magnitude, then clearly

$$\gamma_k = \lambda_k/\lambda_{N+1}\,, \qquad \delta_k = \lambda_k/\lambda_{N+1}\,.$$

From the condition $tg''(t) > 0$ a.e. and conditions (*a*), (*b*), and (*c*) of Theorem *B* we have

$$\lambda_{N-1} < g'(0) \leqslant g'(t) \leqslant \max\{g'(\infty), g'(-\infty)\} < \lambda_{N+1}$$

for all $t \in (-\infty, \infty)$. Hence

$$\lambda_{N-1} < \int_0^1 g'\big(su_0(x)\big)\,ds \leqslant g'\big(u_0(x)\big) < \lambda_{N+1} \tag{39}$$

for all $x \in D$. From (39) and the same type of comparison argument that led to (36) we have the inequalities

$$\alpha_{N-1} < \delta_{N-1} = 1\,, \tag{40}$$

$$1 = \gamma_{N+1} < \alpha_{N+1}\,, \tag{41}$$

$$1 = \gamma_{N+1} < \beta_{N+1}\,. \tag{42}$$

Since $u_0(x) \not\equiv 0$, it follows from (30) that 1 is eigenvalue of (32); hence from (40) and (41) we see that $\alpha_N = 1$. The claim (34) now follows from (42) and (36) with $k = N$.

If

$$V = \operatorname{span}\{\theta_1, \ldots, \theta_N\} \tag{43}$$

then a straight forward calculation shows that

$$\langle v, v\rangle_1 \leqslant \beta_N \int_0 g'\big(u_0(x)\big)\,v^2\,dx \qquad \text{if } v \in V.$$

(Compare with (20)). Therefore, from (23), we conclude that

$$\langle D^2F(u_0)v, v\rangle_1 \leqslant \left(1 - \frac{1}{\beta_N}\right)\langle v, v\rangle_1 \qquad \text{if } v \in V\,. \tag{44}$$

To prove the first assertion of Lemma 3 we show that all of the eigenvalues of the self-adjoint operator $D^2G(x_0)\colon X \to X$ are negative. Assuming the contrary, there exists $h_1 \in X$ such that $\langle D^2G(x_0)h_1, h_1\rangle_1 \geqslant 0$. Let

$$\overline{m} = h_1 + \varphi'(x_0)(h_1)\,. \tag{45}$$

From (12) we have

$$\langle D^2F(u_0)\overline{m}, \overline{m}\rangle_1 \geqslant 0 , \tag{46}$$

and from (11) we see that

$$\langle D^2F(u_0)\overline{m}, k\rangle_1 = 0 \quad \text{ if } k \in Y. \tag{47}$$

Recall, from Lemma 2, that $\langle D^2F(u_0)k, k\rangle_1 \geqslant m\|k\|_1^2$ with $m > 0$ if $k \in Y$. Therefore, by (45), (46), and the self-adjointness of $D^2F(u_0)$, if $k \in Y$ and $\alpha \in R$, then

$$\langle D^2F(u_0)(k + \alpha\overline{m}), k + \alpha\overline{m}\rangle_1 = \langle D^2F(u_0)k, k\rangle_1 + \alpha^2\langle D^2F(u_0)\overline{m}, \overline{m}\rangle_1 \geqslant 0 .$$

Thus, if Z is the subspace of H defined by

$$Z = \{z \in H | z = k + \alpha\overline{m},\ k \in Y,\ \alpha \in R\} , \tag{48}$$

then

$$\langle D^2F(u_0)z, z\rangle_1 \geqslant 0 \quad \text{ if } z \in Z. \tag{49}$$

Extend h_1 to a basis $\{h_1, \ldots, h_N\}$ of X and let

$$\hat{X} = \operatorname{span}\{h_2, \ldots, h_N\} .$$

Since $H = X \oplus Y$, we infer from (45) and (48) that $H = \hat{X} \oplus Z$. Consequently,

$$\theta_k = l_k + z_k , \quad l_k \in \hat{X} , \quad z_k \in Z , \quad 1 \leqslant k \leqslant N.$$

Since $\dim \hat{X} = N - 1$, there exist constants $c_1, \ldots, c_N$ not all zero such that $c_1 l_1 + \ldots + c_N l_N = 0$. Therefore

$$v \equiv c_1\theta_1 + \ldots + c_N\theta_N = c_1 z_1 + \ldots + c_N z_N \in Z ,$$

and $v \neq 0$ by the independence of $\theta_1, \ldots, \theta_N$: By (49) $\langle D^2F(u_0)v, v\rangle_1 \geqslant 0$. On the other hand, from (34) and (44) it follows that $\langle D^2F(u_0)v, v\rangle_1 < 0$. This contradiction shows that all the eigenvalues of $D^2G(x_0)$ must be negative and the first assertion of Lemma 3 is proved.

The second assertion of Lemma 3 follows from the statement that $D^2G(0)$ has one positive eigenvalue and $N - 1$ negative eigenvalues. This statement in turn is equivalent to the conditions: (i) $D^2G(0)$ is nonsingular; (ii) The quadratic form associated with $D^2G(0)$ cannot be negative definite on all of X; (iii) $D^2G(0)$ canno tbe positive definite on any two dimensional subspace of X.

The first condition follows from Lemma 2 since, as shown in the proof of Theorem 4, condition (iii*) of Theorem 4 implies that 0 is a nondegenerate critical point of G.

To establish the second condition assume on the contrary that $\langle D^2G(0)h, h\rangle_1 < 0$ if $h \in X$ and $h \neq 0$. If $\hat{W} = \{h + \varphi'(0)(h) | h \in X\}$ then $\dim \hat{W} = \dim X = N$ and, since $\varphi(0) = 0$, we infer from (12) that $\langle D^2F(0)w, w\rangle_1 < 0$ for $w \neq 0$ and $w \in \hat{W}$. Since the codimension of Y_1 is $N-1$, a standard algebraic argument, which has been used above, shows that there exists $w_1 \in \hat{W} \cap Y_1$ with $w_1 \neq 0$. Therefore $\langle D^2F(0)w_1, w_1\rangle_1 < 0$ and, from Lemma 2 and condition (iv) of Theorem 4, $\langle D^2F(0)w_1, w_1\rangle_1 \geq m_1 \|w_1\|_1^2 > 0$. This contradiction shows that $D^2G(0)$ cannot be negative definite on X.

To prove the third condition suppose that Q is a two-dimensional subspace of X such that $\langle D^2G(0)q, q\rangle_1 > 0$ if $q \in Q$ and $q \neq 0$. If $\hat{Q}$ is the subspace of H defined by

$$\hat{Q} = \{q + \varphi'(0)(q) | q \in Q\}$$

then according to (12), $\langle D^2F(0)\hat{q}, \hat{q}\rangle_1 > 0$ if $\hat{q} \neq 0$ and $\hat{q} \in \hat{Q}$. If $k \in Y$, by (11), $\langle D^2F(0)\hat{q}, k\rangle = 0$. Thus, by condition (viii) of Theorem 4, if $\hat{q} \in \hat{Q}$ and $k \in Y$, then $\langle D^2F(0)(\hat{q}+k), \hat{q}+k\rangle_1 > 0$ unless $\hat{q} = k = 0$; from which we also infer that $\hat{Q} \cap Y = \{0\}$. Since $\operatorname{codimension}(\hat{Q} \oplus Y) = \operatorname{codimension} Y - \operatorname{dimension} \hat{Q} = N-2$ and $\dim X_1 = N-1$ there exists $x_1 \in X_1 \cap (\hat{Q} \oplus Y)$ with $x_1 \neq 0$. From Lemma 2, (iii*) of Theorem 4, and the above $\langle D^2F(0)x_1, x_1\rangle_1 < 0$ and $\langle D^2F(0)x_1, x_1\rangle_1 > 0$. This contradiction proves that $D^2G(x_0)$ cannot have two positive eigenvalues, and by earlier remarks completes the proof of Lemma 3.

As a by-product of the proof of Lemma 3, we have the result that if u_0 is any solution of (P_0) then the boundary value problem

$$\Delta w + g'(u_0(x))\, w = 0 \quad \text{in } D \qquad w|\partial D = 0$$

(in the generalized sense), has only the trivial solution assuming the conditions of Theorem B. This follows from (33) and (34) if u_0 is not identically zero and is a trivial consequence of condition (a) otherwise. By using the same argument, based on the inverse function theorem, that was used in the proof of the second assertion of Theorem A we have

LEMMA 4. – *If u_0 is a solution of (P_0), then, under the conditions of Theorem B, there exist numbers $\delta > 0$ and $\delta' > 0$ such that if $p \in L^2(D)$ and $|p|_0 < \delta$, then there exists a unique weak solution u of (P) with $|u - u_0|_1 < \delta'$.*

LEMMA 5. – *Given $r > 0$ there exists a number $R(r) > 0$ such that if the conditions of Theorem B hold and $p \in L^2(D)$ with $|p|_0 \leq r$ then any weak solution of (P) satisfies $|u|_1 \leq R(r)$.*

PROOF. – If γ and γ' are numbers such that

$$\lambda_N < \gamma' \leqslant \max\{g'(\infty), g'(-\infty)\} \leqslant \gamma < \lambda_{N+1} \tag{50}$$

then according to conditions (*b*) and (*c*) of Theorem *B* there exists a number $t_0 > 0$ such that $\gamma' \leqslant g(t)/t \leqslant \gamma$ if $|t| \geqslant t_0$. We can extend the restriction of $g(t)/t$ to $(-\infty, -t_0] \cup [t_0, \infty)$ (for example, linearly between $-t_0$ and t_0) to a function $h(t)$ continuous on $(-\infty, \infty)$ with

$$\lambda_N < \gamma' \leqslant h(t) \leqslant \gamma < \lambda_{N+1} \tag{51}$$

for all t. Since the function $g(t) - h(t)t$ is continuous and has compact support, it is bounded. Hence

$$g(t) = h(t)\,t + H(t)\,, \qquad |H(t)| \leqslant L \tag{52}$$

for some constant L. Suppose $p \in L^2(D)$ and $|p|_0 \leqslant r$. Let u be a weak solution of (P). If $w \in \mathring{H}_1(D)$ then

$$\int_D \big(\langle \nabla u, \nabla w\rangle - g(u)\,w + p(\xi)\,w\big)\,d\xi = 0\,.$$

Let $u = x + y$ with $x \in X$ and $y \in Y$ and choose $w = y - x$. Since $Y = X^\perp$, $|w|_1 = |u|_1$. From (52) we have

$$\langle y - x, y + x\rangle_1 - \int_D h(u)(y^2 - x^2)\,d\xi = \int_D \Big(H\big(u(\xi)\big)\,w(\xi) - p(\xi)\,w(\xi)\Big)\,d\xi\,;$$

Hence, by (51),

$$|y|_1^2 - \gamma|y|_0^2 + \gamma'|x|_0^2 - |x|_1^2 \leqslant \big(L(\text{meas } D)^{\frac12} + |p|_0\big)|w|_0 \leqslant \big(L(\text{meas } D)^{\frac12} + r\big)\big(1/\sqrt{\lambda_1}\big)|w|_1\,.$$

Therefore, using (19) and (20) we have

$$\left(1 - \frac{\gamma}{\lambda_{N+1}}\right)|y|_0^2 + \left(\frac{\gamma'}{\lambda_N} - 1\right)|x|_1^2 \leqslant \big(L(\text{meas } D)^{\frac12} + r\big)\big(1/\sqrt{\lambda_1}\big)|u|_1\,.$$

Letting $b > 0$ denote the minimum of the two numbers $(1 - \gamma/\lambda_{N+1})$ and $(\gamma'/\lambda_N - 1)$ we see that

$$|u|_1 \leqslant \big(L(\text{meas } D)^{\frac12} + r\big)\big(1/b\sqrt{\lambda_1}\big) \equiv R(r)$$

and the lemma is proved.

LEMMA 6. – *Under the conditions of Theorem B there exist exactly three solutions —one trivial and two nontrivial—of the homogeneous problem* (P_0).

PROOF. – By Lemma 1, it is sufficient to show that there are exactly three solutions of $\nabla G(x) = 0$. If $\nabla G(x) = 0$ then $u = x + \varphi(x)$ is a solution of $\nabla F(u) = 0$ or, equivalently, u is a weak solution of (P_0). Hence, by Lemma 5, $|u|_1 \leqslant R(0)$. Since $\varphi(x) \in Y = X^\perp$, it follows that $|x|_1 \leqslant R(0)$. Now, according to Lemma 3 and the inverse function theorem, the solutions of $\nabla G(x) = 0$ are isolated. Hence, *there can exist only a finite number of solutions of* $\nabla G(x) = 0$. Let $x_1, \ldots, x_k$ denote the nonzero solutions of $\nabla G(x) = 0$; by Theorem A, $k \geqslant 2$. The critical points of G and $f = -G$ coincide, so according to Lemma 3 $\operatorname{sgn}\det D^2 f(0) = (-1)^N \operatorname{sgn}\det D^2 G(0) = -1$, and, if $1 \leqslant i \leqslant k$, $\operatorname{sgn}\det D^2 f(x_i) = (-1)^N \operatorname{sgn}\det D^2 G(x_i) = 1$. By (13), $f(x) \to \infty$ as $|x|_1 \to \infty$ so f satisfies the conditions of Theorem 2. Since $i(\nabla f, x_j) = \operatorname{sgn}\det D^2 f(x_j)$, it follows from Theorem 2 and the above that $-1 + k = 1$. Hence $k = 2$ and the Lemma is proved.

To complete the proof of Theorem B let u_0, u_1, and u_2 be the three solutions of the problem (P_0). According to Lemma 4, we can choose numbers $\delta > 0$ and $\delta' > 0$ such that if $p \in L^2(D)$ and $|p|_0 < \delta$ then there exist solutions $\tilde{u}_k$, $k = 0, 1, 2$ of problem (P) with $|\tilde{u}_k - u_k|_1 < \delta'$ for $k = 0, 1, 2$. Since δ and δ' can be taken to be arbitrarily small, these solutions will be distinct if $|p|_0$ is small.

Assuming that Theorem B is not true, there exists a sequence $\{p_m\}_1^\infty$ in $L^2(D)$ such that $|p_m|_0 \to 0$ as $m \to \infty$ and such that there exist four distinct solutions u_{lm}, $l = 0, 1, 2, 3$, of (P) with $p = p_m$. Consequently, if $w \in \mathring{H}_1(D)$ and $0 \leqslant l \leqslant 3$, then

$$\int_D \big(\langle \nabla w, \nabla u_{lm}\rangle - g(u_{lm})\, w + w p_m\big)\, d\xi = 0 \tag{53}$$

If $N: \mathring{H}_1(D) \to \mathring{H}_1(D)$ is defined by $\langle N(u), w\rangle_1 = \int_D g(u)\, w\, d\xi$ then since g has a bounded derivative and the injection $\mathring{H}_1(D) \to L_2(D)$ is compact ([4]), N is continuous with respect to weak convergence. From the Riesz representation theorem there exists $v_m \in \mathring{H}_1(D)$ such that $\langle w, p_m\rangle_0 = \langle w, v_m\rangle_1$ and, since

$$|v_m|_1^2 \leqslant |p_m|_0 |v_m|_0 \leqslant |p_m|_0 \big(1/\sqrt{\lambda_1}\big) |v_m|_1\,, \qquad |v_m|_1 \to 0 \text{ as } m \to \infty.$$

We can thus write (53) in the form

$$\langle u_{lm}, w\rangle_1 = \langle N(u_{lm}), w\rangle_1 - \langle v_m, w\rangle_1$$

for all $w \in \mathring{H}_1(D)$; hence

$$u_{lm} = N(u_{lm}) - v_m\,, \qquad 0 \leqslant l \leqslant 3 \tag{54}$$

Since the sequence $\{p_m\}_1^\infty$ is bounded in $L^2(D)$, it follows from Lemma 5 that the sequences $\{u_{lm}\}_{m=1}^\infty$ are bounded in $\mathring{H}_1(D)$. Hence, there is a sequence of integers $\{m_j\}_{j=1}^\infty$ such that $\{u_{lm_j}\}_{j=1}^\infty$ converges weakly to some z_l in $\mathring{H}_1(D)$ for $0 \leqslant l \leqslant 3$. Since $N(u_{lm_j})$ converges strongly to $N(z_l)$ for $l = 0, 1, 2, 3$, it follows from (54) that

$\{u_{lm_j}\}_1^\infty$ actually converges strongly to z_l and $z_l = N(z_l)$ for $0 \leqslant l \leqslant 3$. Hence, for $w \in \mathring{H}_1(D)$, $\langle z_l, w\rangle_1 = \langle N(z_l), w\rangle_1$ or $\int_D (\langle \nabla z_l, w\rangle - g(z_l) w)\, d\xi = 0$ if $l = 0, 1, 2, 3$, so z_l is a weak solution of (P_0). Therefore, each z_l, $l = 0, 1, 2, 3$, is equal to some u_k, $k = 0, 1, 2$. This means that some two of the four sequences $\{u_{lm_j}\}_{j=1}^\infty$, $0 \leqslant l \leqslant 3$ must converge to the same weak solution of (P_0). Since for each j, the four functions u_{lm_j}, $l = 0, 1, 2, 3$ are distinct, and since $|p_{m_j}|_0 \to 0$ as $j \to \infty$, this contradicts Lemma 4 for j large and Theorem B is proved.

5. – Proof of Theorem C.

To prove Theorem C we make use of the following result due to CLARK [8, p. 71] which was actually stated in the more general context of C^1 functions defined on Banach spaces which have a second derivative only at the origin:

Let H be a real Hilbert space and f an even, real-valued C^2 function defined on H. Suppose that f has the property that whenever $\{x_n\} \subseteq H$ is a bounded sequence such that $f(x_n) < 0$, $f(x_n)$ is bounded below, and $\nabla f(x_n) \to 0$, then $\{x_n\}$ contains a convergent subsequence. Suppose that $f(0) = 0$, f is bounded below, there exists a subspace M of H of dimension $l > 0$ such that $\langle D^2 f(0)x, x\rangle < 0$ if $x \in M$ with $x \neq 0$, and $f(x) \geqslant 0$ for $|x|$ sufficiently large. Then there exist at last $2l$ nonzero solutions of $\nabla f(x) = 0$.

To prove Theorem C we again define $F\colon \mathring{H}_1(D) \to R$ by (22) and observe that, since g is odd, F is even. As shown in the proof of Theorem A, F satisfies the hypotheses of Theorem 4. We assert that the function φ defined by (7) and (8) is odd. Indeed, by the oddness of ∇F, it follows from (7) that $\langle \nabla F(-x - \varphi(x)), k\rangle = -\langle \nabla F(x + \varphi(x)), k\rangle = 0$ for all $k \in Y$. Since $\varphi(-x)$ is the unique element of Y such that $\langle \nabla F(-x + \varphi(-x)), k\rangle = 0$ for all $k \in Y$, it follows that $\varphi(-x) = -\varphi(x)$. From this it follows that the function $G\colon X \to R$ defined by $G(x) = F(x + \varphi(x))$ is also even.

We assert that, under the conditions of Theorem C, the quadratic form associated with $D^2G(0)$ is positive definite on some subspace M of X of dimension $N - K + 1$. Assuming the contrary, $D^2G(0)$ has at least K nonpositive eigenvalues so there exists a subspace W of X with $\dim W \geqslant K$ such that $\langle D^2G(0)w, w\rangle \leqslant 0$ for all $w \in W$. If $\hat{W}$ is the subspace of $\mathring{H}_1(D)$ defined by $\hat{W} = \{w + \varphi'(0)w | w \in W\}$, then, since $\varphi(0) = 0$, it follows from (12) that $\langle D^2F(0)v, v\rangle \leqslant 0$ for all $v \in \hat{W}$. By (21) and the hypotheses of Theorem C, it follows that if Y_1 is defined as in the proof of Theorem A, then there exists $m_1 > 0$ such that $\langle D^2F(0)s, s\rangle \geqslant m_1\langle s, s\rangle$ for all $s \in Y_1$ and codimension $Y_1 = K - 1$. Since $\dim \hat{W} = \dim W \geqslant K$ there exists $z \in \hat{W} \cap Y_1$ with $z \neq 0$ which is clearly impossible. This contradiction establishes the existence of an $(N - K + 1)$-dimensional subspace M of X on which $D^2G(0)$ is positive definite.

If $f(x) = -G(x)$ then $\langle D^2 f(0)x, x\rangle < 0$ if $x \in M$ and $x \neq 0$. Moreover, by (13), $f(x) \to +\infty$ as $|x| \to \infty$. Since $\dim X = N < \infty$, it is clear that $f\colon X \to R$ satisfies all of the conditions of the aforementioned result due to Clark. This establishes the existence of at least $2(N - K + 1)$ nonzero solutions of $\nabla G(x) = 0$, and so by Lemma 1 of the previous section, there exist at least $2(N - K + 1)$ nonzero solutions of $\nabla F(u) = 0$. Since critical points of F are weak solutions of (P_0) this proves Theorem C.

6. – Regularity.

The fact that, under the hypotheses of Theorem A, any weak $\mathring{H}_1$-solution of (P) is also a classical solution follows from a standard « bootstrap » argument which we indicate for completeness.

Let $W_{m,p}(D)$ denote the standard Sobolev space of functions having generalized $L^p(D)$-derivatives up to order $m \geq 1$. If u is a weak solution of (P) in $\mathring{H}_1(D) \subset W_{1,2}$ then $u \in L^2(D)$. Assume that we have established that $u \in L^q(D)$ for some $q \geq 2$. Since g' is bounded and $p(x) \in C^\alpha(D)$, for some $\alpha \in (0, 1)$,

$$h(x) \equiv p(x) - g\big(u(x)\big) \in L^q(D)\,.$$

Therefore, since u is a generalized solution of

$$\begin{aligned} (\Delta u)(x) &= h(x)\,, \qquad x \in D \\ u(x) &= 0\,, \qquad x \in \partial D\,, \end{aligned}$$

if follows from a result due to AGMON, DOUGLIS and NIRENBERG [1] that $u \in W_{2,q}$: If $2q \geq n$ then by the Sobolev imbedding theorem (see [4, p. 221]) it follows that $u \in L^r(D)$ for any $r \in [1, \infty)$. If $2q < n$ then $u \in L^r(D)$ where $r = nq/(n - 2q) > q$. Repeating this argument a sufficient number of times, we can conclude that $u \in W_{2,r}$ where r is so large that $0 < \alpha < 1 - (n/r)$. Since $W_{2,r} \subseteq W_{1,r}$, it follows from a result due to MORREY [19, Theorem 3.3.3] that $u \in C^\beta(D)$ where the Hölder exponent $\beta = 1 - (n/r)$. Since g' is continuous and $C^\beta(D) \subseteq C^\alpha(D)$, $h(x) = p(x) - g\big(u(x)\big) \in C^\alpha(D)$. Therefore, since ∂D is of class $C^{2+\alpha}$, it follows by standard potential theory [10] that there exists a solution of

$$\begin{aligned} \Delta(v)(x) &= h(x)\,, \qquad x \in D \\ v(x) &= 0\,, \qquad x \in \partial D\,, \end{aligned} \tag{55}$$

with $v \in C^{2+\alpha}$. Since u and v are both $\mathring{H}_1$-weak solutions of (53) and (55) has a unique $\mathring{H}_1$-weak solution, $u = v$. This proves that u is a classical solution of (P).

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